CONSERVATIVE NUMERICAL SCHEMES FOR THE RADIAL VLASOV-POISSON SYSTEM: STABILITY AND LANDAU DAMPING

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Abstract. In this paper, we build numerical conservative schemes for the radial Vlasov-Poisson system in order to observe the behavior of solutions around steady states. These schemes are based on finite differences method and provide the conservation of the mass (L^1 norm) and the Hamiltonian. To assure these properties and the convergence of the schemes we treat in particular the problem of singularities linked to the radial geometry.

For the moment, this first version of this paper give the expression of the schemes and proves the conservational properties The next version, coming very soon, will add the visualization of the stable behavior and of the Landau Damping phenomenon.

1. Introduction

The gravitational Vlasov-Poisson equation is a very well known stellar model which describes the motion of a self-graviting system. In the general 3D case, it takes the form

$$\left\{ \begin{array}{l} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ \\ f(t=0,x,v) = f_0(x,v) \geq 0, \end{array} \right. \tag{1.1}$$

where the gravitational potential ϕ_f satisfies

$$\left\{ \begin{array}{l} \triangle \phi_f(x) = \rho_f(x) = \int_{\mathbb{R}^3} f(x,v) dv, \\ \phi_f(x) \to 0 \text{ quand } |x| \to +\infty. \end{array} \right. \tag{1.2}$$

Our aim is this paper is to discuss trough numerical results about stability of stationary solutions for this system. In the past decade, our knowledge in this domain improves greatly. In one hand, in a toric space, Villani and Mouhot [3] proved that Landau Damping holds around all stationary solutions homogeneous in the velocity variable v. Recently, Bedrossian, Masmoudi and Mouhot [1] gave a new, simpler, proof for this result. In the other hand, in the entire $\mathbb{R}^3 \times \mathbb{R}^3$ space, M. Lemou, F. Méhats and P. Raphaël [4] proved the orbital stability of a very large class of stationary solutions. However, in this case, the question of the Landau Damping stays open.

In this paper we do not propose any theorical result but we build conservative numerical schemes, which could be used to better understand the phenemenom of Landau Damping for the Vlasov-Poisson system.

To avoid the complexity corresponding to the 7 dimensions of the system (1.1), we choose to restrain our study to the Vlasov-Poisson system in the radial coordinates $(|x|, |v|, |x \cdot v|)$. This choice does not seems too restritive: indeed, all class of stationary solutions which has been studied in previous articles are radial, up to a translation shift in space.

Moreover we gain 3 dimensions which gives a set of work Ω defined by

$$\Omega = \{(r, u, s) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R},\ |s| \leq ru\},$$

where the variables (r, u, s) correspond to $(|x|, |v|, |x \cdot v|)$. Hence, in radial coordinates, the Vlasov-Poisson equation takes the form

$$\partial_t f + \frac{s}{r}\partial_r f - \frac{s}{ru}\phi_f'(r)\partial_u f + \left(u^2 - r\phi_f'(r)\right)\partial_s f = 0, \quad (t, r, u, s) \in \mathbb{R}_+ \times \Omega, \quad (1.3)$$

and the expressions of the gravitational force and the density can be rewrited

$$\phi_f'(r) = \frac{1}{r^2}\int_{\tilde{r}<r} \tilde{r}^2 \rho_f(\tilde{r})d\tilde{r} \quad \text{and} \quad \rho_f(r) = \frac{2\pi}{r}\int_{u>0,\ |s|<ru} f(r, u, s)ududs <; \quad (1.4)$$

Remark that both equations (1.3) and (1.4) are not equivalent to the Vlasov-Poisson system (1.1): indeed the set Ω has a typical form with a border

$$\partial\Omega = \{(s, r, u) \in \Omega, s = ru\},$$

on which the border condition is

$$\nabla f \cdot \overrightarrow{n} = 0,$$

where $\overrightarrow{n}$ is the normal to the border $\partial\Omega$. Then we have our complete system of equations and we aim to obtain a numerical scheme to visualize their solutions.

First let talk about our numerical approach and about the type of method that we will use. Our purpose is to obtain numerical conservative schemes mainly for three reasons. The first one that some studies treat numerically this equation ([5] with PIC methods, [2] with operator Splitting methods) but none of them preserved the L^1-norm and the Hamiltonien. Second it would imply the robustness of our scheme and third it would be well adapted with the theorical studies which strongly use the rigidity of the flow.

Hence in order to obtain this conservation we will choose numerical methods based on finite diffenrences. Since the conservation of the mass takes the form

$$M(f(t)) := 8\pi^2 \int_\Omega ruf(t, r, u, s)drduds = M(f_0),$$

every finite differences scheme will give the preservation of the mass in the inner of Ω. However, from the form of Ω, it seems rather complated to obtain a preserving one's near $\partial\Omega$.

To avoid this difficulty, we proceed to a change of variables. Some new sets of variables are possible, as for exemple

$$(r, u, \theta) = (|x|, |v|, \arccos(\frac{x \cdot v}{|x||v|})) \quad \text{or} \quad (r, q, l) = (|x|, \frac{x \cdot v}{|x|}, |x \wedge v|),$$

but for all of them, a better form of domain corresponds to the appearance of singularities in the equation. Hence, since the form of domain and the singularities are linked (which is evident), we are going to take the best equilibrium between these both difficulties.

Note that a complexe form of domain is not as bad as it seems if the dimension of the space is small. Thus, a first idea is to take as variable the invariants of the transport operator

$$\frac{s}{r}\partial_r f - \frac{s}{ru}\phi_f'(r)\partial_u f + \left(u^2 - r\phi_f'(r)\right)\partial_s f,$$

which are completely known for the radial case [?]: they are the microscopic energy and the square of the kinetic momentum

$$e = \frac{|v|^2}{2} + \phi_f(x) \quad \text{and} \quad L = |x \wedge v|^2 = r^2u^2 - s^2.$$

However, with these variables, an other problem appears: the time-dependance of the domain (since the microscopic energy is not constant in time), which could become an even bigger difficulty. To avoid it we remplace the our microscopic energy variable e by the variable $w = u^2/2$.

Our new set of variable is finally

$$(r, w, L) = \left(sgn(s)r, \frac{u^2}{2}, r^2u^2 - s^2\right) = \left(sgn(x \cdot v)|x|, |v|, |x \wedge v|^2\right),$$

where sgn is the sign fonction. The prolongement of the variable r to all $\mathbb{R}$ is necessary since the variable $L = |x \wedge v|^2$ does not take into account the signe of $s = x \cdot v$. The domain of study is then

$$\Gamma = \{(r, w, L) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+,\ L \leq 2r^2w\},$$

and the kinetic equation (1.3) gives

$$\partial_t f + \sqrt{2\left(w - \frac{L}{2r^2}\right)}\partial_r f - \phi_f'(r)\sqrt{2\left(w - \frac{L}{2r^2}\right)}\partial_w f = 0, \tag{1.5}$$

where the gravitational field of force ϕ_f' and the density ρ_f are defined by

$$\begin{aligned} &\text{for all } r \in \mathbb{R}^*, \quad \phi_f'(r) = \frac{1}{r^2}\int_{-r}^{r} \tilde{r}^2 \rho_f(\tilde{r})d\tilde{r}, \\ &\text{for all } r \in \mathbb{R}^*, \quad \rho_f(r) = \frac{2\pi}{r^2}\int_0^{+\infty} dw \int_0^{2r^2w} \frac{f(r, w, L)}{\sqrt{2\left(w - \frac{L}{2r^2}\right)}}dL. \end{aligned} \tag{1.6}$$

Note that we have decompose the real density which corresponds to

$$\rho_f(r) + \rho_f(-r),$$

which explains the expression of ϕ_f' : here this field of forces is just the even prolongement of the real ϕ_f'. Moreover this change of variable brings some border conditions: indeed if we decompose our domain such that

$$\Gamma = \bigcup_{L \geq 0} \Gamma_L = \bigcup_{L \geq 0} \left\{(r, w, L) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+,\ 2r^2w \geq L\right\}, \tag{1.7}$$

the border domain $\partial\Gamma$ appears in a graphic representation of Γ_L (see figure 1) Then the border conditions corresponds to the continuity of the distribution function f and the conservation of the mass passing from one zone the other: from $\{r < 0\}$ to

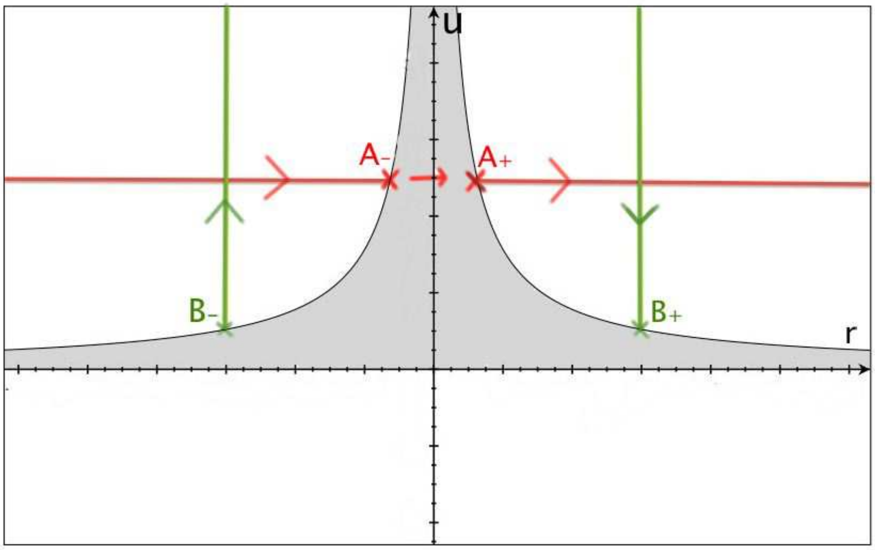

FIGURE 1. Sens of transport in the characteristic plan Γ_L for $L > 0$.

$\{r > 0\}$ or the inverse and since Γ_L is a characterisitic plan (L is an invariant for the transport operator), these conditions can be mathematicaly writed as

$$f(-r, w, 2r^2 w) = f(r, w, 2r^2 w) \tag{1.8}$$

$$\lim_{r\sqrt{2w}\to -\sqrt{L}^-} \sqrt{2\left(w - \frac{L}{2r^2}\right)} \nabla f \cdot \mathbf{n}_- = - \lim_{r\sqrt{2w}\to -\sqrt{L}^+} \sqrt{2\left(w - \frac{L}{2r^2}\right)} \nabla f \cdot \mathbf{n}_+, \tag{1.9}$$

where $\mathbf{n}_+$ and $\mathbf{n}_-$ are the outgoing normals to $\partial\Omega$ respectively at $r = -\sqrt{L/2w}$ and at $r = \sqrt{L/2w}$. the second condition (1.9) implies en particular

$$\left\{ \begin{array}{l} \displaystyle \lim_{r\sqrt{2w}\to -\sqrt{L}^-} \sqrt{2\left(w - \frac{L}{2r^2}\right)} \partial_r f = \lim_{r\sqrt{2w}\to -\sqrt{L}^+} \sqrt{2\left(w - \frac{L}{2r^2}\right)} \partial_r f, \\ \\ \displaystyle \lim_{r\sqrt{2w}\to -\sqrt{L}^-} \sqrt{2\left(w - \frac{L}{2r^2}\right)} \partial_w f = - \lim_{r\sqrt{2w}\to -\sqrt{L}^+} \sqrt{2\left(w - \frac{L}{2r^2}\right)} \partial_w f. \end{array} \right. \tag{1.10}$$

If we look at the figure 1, these conditions correspond to the transfert of the mass from A^- to A_+ and from B^+ to B^-.

Remark that the term $\sqrt{2\left(w - \frac{L}{2r^2}\right)}$ converges to 0 on the border domain. But it is necessary to add it in the condition (1.10) to compensate a possible blow-up of the gradiant ∇f. (this blow-up could appear even if f is a $\mathcal{C}^1$ function on $\mathbb{R}^6$.)

To complete this model and before considering its numerical study, we give the expression of the mass and the Hamiltonian with these variables:

$$M(f(t)) = 8\pi^2 \int_\Gamma \frac{f(r, w, L)}{\sqrt{2\left(w - \frac{L}{2r^2}\right)}} drdwdL = M_l(f_0), \tag{1.11}$$

and

$$H(f(t)) = E_{kin}(f(t)) - E_{pot}(f(t)) = H(f_0), \tag{1.12}$$

where the kinetic energy $E_{kin}(f)$ and the potential energy $E_{pot}(f)$ are defined by

$$\begin{cases} E_{kin}(f) = 8\pi^2 \displaystyle\int_\Gamma \frac{wf(r,w,L)}{\sqrt{2\left(w-\frac{L}{2r^2}\right)}} drdwdL \\ \\ E_{pot}(f) = 2\pi \displaystyle\int_0^{+\infty} r^2 \left[\phi_f'(r)\right]^2 dr. \end{cases} \tag{1.13}$$

In the next section, we propose a numerical scheme which preserved the mass and the Hamiltonian.

2. Numerical conservative schemes for the radial VP system

2.1. **Discretisation and notations.** At fixed $T > 0$, we choose classic discretisations $(t_0 = 0, ..., t_n, ..., t_N = T)$ with $t_n = n\Delta t$, and $(r_k)_{k\geq 1}$ and $(w_i)_{i\geq 1}$ such that

$$\begin{aligned} &\text{for } k \geq 1, \quad r_k = \left(k - \tfrac{1}{2}\right)\Delta r \quad \text{and} \quad r_{-k} = -r_k, \\ &\text{for } i \geq 1, \quad w_i = \left(i - \tfrac{1}{2}\right)\Delta w. \end{aligned} \tag{2.1}$$

Note that the time steps Δt, Δr and Δ_w are constant. Futhermore, since it L is an invariant of the transport operator, we can take a variable step for L as small as we want (there will not be CFL condition on it) : thus, we assure the non emptiness of the sets

$$\{L_j,\ L_j < 2r_k^2 w_i\},\ \text{at fixed } r_k \text{ and } w_i,$$

and a good approximation of the distribution for r and u small. Hence, for exemple, by defining the subdivision $(\overline{L}_j)_{j\geq 0}$ corresponding to

$$0 < \frac{\Delta L}{2^p} < \frac{\Delta L}{2^{p-1}} < ... < \frac{\Delta L}{2} < \Delta L < 2\Delta L < 3\Delta L < ... \tag{2.2}$$

we could considere the following centred discretisation with respect of L:

$$\text{for all } j \geq 0, \quad L_j = \frac{\overline{L}_j + \overline{L}_{j+1}}{2}. \tag{2.3}$$

It completes our system of discretisations. Now, let pass to the definition of our numerical schem. We note $(f_{k,i,j}^n)$ the classical approximation

$$f_{k,i,j}^n \simeq f(t_n, r_k, w_i, L_j)$$

of the distribution function f, solution of the kinetic equation

$$\partial_t f + \sqrt{2w - \frac{L}{r^2}}\partial_r f - \phi_f'(r)\sqrt{2w - \frac{L}{r^2}}\partial_w f = 0. \tag{2.4}$$

on Γ with the border conditions (1.8) and (1.10). Then this approximation will satisfies a numerical schem of the general form

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$$\textbf{Global scheme :}\quad D_t f^n = \frac{\tilde{f}^{n+1} - \tilde{f}^n}{\Delta t} = -\alpha_{k,i,j} D_r f^n + \left[\phi_{f^n}'\right]_{r_k} \alpha_{k,i,j} D_w f^n, \tag{2.5}$$

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where D_r et D_u are numerical diffential operators and the terms $\left[\phi'_{f^n}\right]_{r_k}$ and $\alpha_{k,i,j}$ are approximations of respectively, ϕ'_{f^n} and $\sqrt{2w-\frac{L}{r^2}}$. In the expression of the differential operator D_t, the terms $\tilde{f}^n$ are approximations of f^n and their expressions provide the order of the operator D_t.

To simplify the notation in the following study, for an differential operator D and an identity operator $(f)\mapsto(\tilde{f})$, we will not their differential adjoint operators, respectively D^* and $(f)\mapsto(\tilde{f}^*)$, which satisfy

$$\sum g\Delta f=\sum f\Delta^* g \quad \text{and} \quad \sum \tilde{f}g=\sum f\tilde{g}^*.$$

Finally, to complete the notations, we note $m_{k,i,j}$, the local mass, which corresponds to the following definitions:

Definition 2.1. *From the distribution function $(f^n_{k,i,j})$, we define numerically the density (ρ^n_k), the mass M^n, the gravitational field (E^n_k) and the Hamiltonian H^n by*

- $\forall k\in\mathbb{N}^*,\quad \rho^n_k=\dfrac{2\pi}{r_k^2}\displaystyle\sum_{\substack{i,j\\ \overline{L}_{j+1}<2r_k^2w_i}} m_{k,i,j}f^n_{k,i,j}\Delta w(\Delta L)_j,$
- $M^n=4\pi\displaystyle\sum_{k=1}^{+\infty} r_k^2\left(\rho^n_k+\rho^n_{-k}\right)\Delta r=8\pi^2\sum_{\substack{k,i,j\\ \overline{L}_{j+1}<2r_k^2w_i}} m_{k,i,j}f^n_{k,i,j}\Delta r\Delta u(\Delta L)_j,$
- $E^n_k=\dfrac{1}{r_k^2}\displaystyle\sum_{\tilde{k}=-k+1}^{k} I_{\tilde{k}}(r^2\rho^n)\Delta r,$ *where* $I_{\tilde{k}}(g)\Delta r\simeq\int_{r_{k-1}}^{r_k} g\,dr,$
- $H^n=E^n_{cin}-E^n_{pot}$, *avec*

$$\left\{\begin{aligned} E^n_{cin}&=8\pi^2\sum_{\substack{k,i,j\\ \overline{L}_{j+1}<2r_k^2w_i}} w_i m_{k,i,j}f^n_{k,i,j}\Delta r\Delta u(\Delta L)_j\\ E^n_{pot}&=2\pi\sum_{k=1}^{+\infty} r_k^2\left(E^n_k\right)^2\Delta r.\end{aligned}\right. \tag{2.6}$$

In the next part, we precise this notation, which are very general for the moment.

2.2. **Conservative numerical schemes.** We propose here to give the expression of all the quantities define in the previous part. This expression will come from our purpose to have conservative and converging schems.

2.2.1. *First order expansion with respect of the kinetic momentum L and conservation of the mass.* Let first precise the local mass $m_{k,i,j}$. At fixed r and w non zeros, and for $\overline{L}_{j+1}<2r^2w$, a first order Taylor expansion of f around L_j provides

$$\int_{\overline{L}_j}^{\overline{L}_{j+1}}\frac{f}{\sqrt{2w-\frac{L}{r^2}}}dL=2r^2\left(\sqrt{2w-\frac{\overline{L}_j}{r^2}}-\sqrt{2w-\frac{\overline{L}_{j+1}}{r^2}}\right)\left(f(r,w,L_j)+\left((\Delta L)_j\right)\right).$$

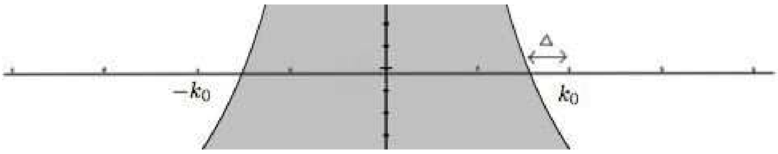

FIGURE 2. Discretisation at fixed $w > 0$ and $L > 0$

Thus we define $m_{k,i,j}$ by

$$m_{k,i,j}(\Delta L)_j = 2r^2 \left(\sqrt{2w_i - \frac{\overline{L}_j}{r_k^2}} - \sqrt{2w_i - \frac{\overline{L}_{j+1}}{r_k^2}} \right). \tag{2.7}$$

With the same approach, we approximate the terms $\sqrt{2w - \frac{L}{r^2}}$ in (2.4) by

$$\alpha_{k,i,j} = \frac{1}{2} \left(\sqrt{2w_i - \frac{\overline{L}_j}{r_k^2}} + \sqrt{2w_i - \frac{\overline{L}_{j+1}}{r_k^2}} \right), \tag{2.8}$$

which implies $m_{k,i,j}\alpha_{k,i,j} = 1$ for all (k, i, j). By supposing that D_r and D_u are independant of the variable L_j, we obtain then the conservation of the mass. Let detail this property for the operator D_r. At fixed $w > 0$ and $L > 0$, let watch the figure 2. To have the stability of the scheme, for $k \neq k_0$, we choose an upwind scheme of the form

$$D_r f_k = \frac{\overline{f}_k - \overline{f}_{k-1}}{\Delta r},$$

where $(f) \mapsto (\overline{f})$ is an identity operator. This operator is consistant. To assure the conservation of the mass, we have then to take in $k = k_0$

$$D_r f_{k_0} = \frac{\overline{f}_{k_0} - \overline{f}_{-k_0}}{\Delta r}, \tag{2.9}$$

instead of the expression $D_r f_{k_0} = \frac{\overline{f}_{k_0} - \overline{f}_{-k_0}}{2\Delta}$ which seems consistant. In reality, to study consistance, we have to study both terms α and D_r together and not separately. Indeed in the theorical model and near the border $\partial_r f$ blows up and α goes to 0. Thus, in fact, the conservation of the mass brings consistance. Hence the definitions of the operator D_r and D_w follows:

Definition of the differential operators D_r and D_w

For (k, i, j) such that $\overline{L}_{j+1} < 2r_k^2 w_i$, we note

$$\begin{cases} \forall i > 0, \quad D_{r_k} f^n = \dfrac{\overline{f}_k^n - \overline{f}_{k-1}^n}{\Delta r}, \quad \text{if } \; r_{k-1} > \sqrt{\dfrac{\overline{L}_{j+1}}{2w_i}} \quad \text{or if } \; r_k < -\sqrt{\dfrac{\overline{L}_{j+1}}{2w_i}}, \\[2ex] \forall k > 0, \quad D_{w_i} f^n = \dfrac{\hat{f}_{i+1}^n - \hat{f}_i^n}{\Delta w}, \quad \text{if } \; w_i > \dfrac{\overline{L}_{j+1}}{2r_k^2}, \\[2ex] \forall k < 0, \quad D_{w_i} f^n = \dfrac{\hat{f}_i^n - \hat{f}_{i-1}^n}{\Delta w}, \quad \text{if } \; w_{i-1} > \dfrac{\overline{L}_{j+1}}{2r_k^2}, \end{cases} \tag{2.10}$$

with the border conditions,

$$\begin{cases} \forall i > 0, \quad D_{r_k} f^n = \dfrac{\overline{f}^n_k - \overline{f}^n_{-k}}{\Delta r} & \text{if } \ r_{k-1} \leq \sqrt{\dfrac{\overline{L}_{j+1}}{2 w_i}} < r_k, \\[2ex] \forall k < 0, \quad D_{w_i} f^n = \dfrac{\hat{f}^n_{-k,i} - \hat{f}^n_{k,i}}{\Delta w} & \text{if } \ w_{i-1} \leq \dfrac{\overline{L}_{j+1}}{2 r_k^2} < w_i. \end{cases} \tag{2.11}$$

where the application $(f) \mapsto (\overline{f})$ and $(f) \mapsto (\hat{f})$ are approximation of the identity which we choose in function of the desired order of the differential operators.

With these definition, the scheme (2.5) isconservative. Moreover to assure its convergence, we need, of course, that a CFL condition

$$\sqrt{2 w_i} \frac{\Delta t}{\Delta r} + \left| \left[\phi'_{f^n} \right]_{r_k} \right| \frac{\Delta t}{\Delta w} < 1. \tag{2.12}$$

is satisfied. Now let give the expression of the field $\left[\phi'_{f^n} \right]_{r_k}$ to obtain the conservation of the Hamiltonian.

2.2.2. *Approximation of the gravitational field for the conservation of the Hamiltonian.* We begin by analysing the variation of the potential energy given in the Definition 2.1:

$$E^n_{pot} = 2\pi \sum_{k=1}^{+\infty} r_k^2 \left(E^n_k \right)^2 \Delta r,$$

where the gravitational field E^n_k is even and satisfies

$$\text{for all } k > 0, \quad r_k^2 E^n_k = \sum_{\tilde{k}=-k+1}^{k} I_{\tilde{k}}(r^2 \rho^n) \Delta r.$$

We first remark that

$$D_t \left(E^n_{pot} \right) = \frac{\tilde{E}^{n+1}_{pot} - \tilde{E}^n_{pot}}{\Delta t} = 4\pi \sum_{k=1}^{+\infty} D_t \left(r_k^2 E^n_k \right) \frac{\tilde{E}^{n+1}_k + \tilde{E}^n_k}{2} \Delta r. \tag{2.13}$$

Furthermore, to determine the variation of $D_t \left(r_k^2 E^n_k \right)$, we notice from the previous study on the mass conservation that

$$\begin{aligned} D_t(r_k^2 \rho^n_k) &= 2\pi \sum_{\substack{i,j \\ \overline{L}_{j+1} < 2 r_k^2 w_i}} m_{k,i,j} D_t(f^n_{k,i,j}) \Delta w (\Delta L)_j, \\ &= -2\pi \sum_{\substack{i,j \\ \overline{L}_{j+1} < 2 r_k^2 w_i}} \frac{\overline{f}^n_k - \overline{f}^n_{k-1}}{\Delta r} \Delta w (\Delta L)_j, \end{aligned}$$

where we use the numerical scheme (2.5). Thus we obtain

$$\begin{aligned} D_t\left(r_k^2 E_k^n\right) &= -2\pi \sum_{\tilde{k}=-k+1:k} \sum_{\substack{i,j \\ \overline{L}_{j+1}<2r_k^2 w_i}} I_{\tilde{k}}(\overline{f}_k^n - \overline{f}_{k-1}^n)\Delta w(\Delta L)_j, \\ &= -2\pi \sum_{\substack{i,j \\ \overline{L}_{j+1}<2r_k^2 w_i}} \left[I(\overline{f}^n)_k - I(\overline{f}^n)_{-k}\right]\Delta w(\Delta L)_j, \end{aligned} \tag{2.14}$$

and finally, by noting $I(\overline{f}) = \overline{I}(f)$ and $\overline{I}^*$ its adjoint operator,

$$D_t\left(E_{pot}^n\right) = -8\pi^2 \sum_{\substack{k,i,j \\ \overline{L}_{j+1}<2r_k^2 w_i}} f_{k,i,j}^n \overline{I}^* \left(\frac{\tilde{E}_k^{n+1}+\tilde{E}_k^n}{2}\right)\Delta r\Delta u(\Delta L)_j. \tag{2.15}$$

Now let study the variation of the kinetic equation given in the Definition 2.1:

$$E_{cin}^n = 8\pi^2 \sum_{\substack{k,i,j \\ \overline{L}_{j+1}<2r_k^2 w_i}} w_i m_{k,i,j} f_{k,i,j}^n \Delta r\Delta u(\Delta L)_j$$

From the expression of the numerical scheme (2.5), we have

$$D_t\left(E_{cin}^n\right) = 8\pi^2 \sum_{\substack{k,i,j \\ \overline{L}_{j+1}<2r_k^2 w_i}} D_w^*(w_i)\left[\phi'_{f^n}\right]_{r_k} f_{k,i,j}^n \Delta r\Delta u(\Delta L)_j,$$

and, if D_w is at least an first order differential operator, $D_w^*(w_i) = -1$ and we obtain the condition

$$\left[\phi'_{f^n}\right]_{r_k} = \overline{I}^* \left(\frac{\tilde{E}_k^{n+1}+\tilde{E}_k^n}{2}\right), \tag{2.16}$$

for the conservation of the Hamiltonian. Instead of the $\tilde{E}_k^{n+1}$ term, this schem saty an explicite one because this term can be directly calculated from the equation (2.14).

3. Numerical tests

This part will be developped in the months of mars 2014.

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